Section 5.2
Confidence Intervals for a Proportion

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- In this section we discuss the notion of an **interval estimate** for a population parameter.
- A plausible range of values for the population parameter is called a *confidence interval*.

Confidence intervals

- Using only a sample statistic to estimate a parameter is like fishing in a murky lake with a spear, and using a confidence interval is like fishing with a net.
- If we report a point estimate, we probably won't hit the exact population parameter. If we report a range of plausible values we have a good shot at capturing the parameter.
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- For instance, you might read in the paper

  A 95% confidence interval for the proportion of voters in favor of this issue is $57\% \pm 3\%$.

  In other words, the interval tells us that, based on our sample, we believe the population proportion p is somewhere between 54% and 60%.

A 95% Confidence Interval for p

Formula

A 95% confidence interval for a population proportion p, based on a sample of size n that produced sample proportion $\hat{p}$, is given by the formula:

$$\hat{p} \pm 1.96 \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}.$$

Yellow Rectangles

Estimate the population proportion p of yellow rectangles in Hitchman's rectangle data set.

- We sample $n = 100$, and find 23 of the 100 rectangles are yellow (Dane is not among them).
- $\hat{p} = 23/100 = 0.23$ (the center of our interval)
- The margin of error (MOE):

$$\begin{aligned} \text{MOE} &= 1.96 \cdot \sqrt{\hat{p}(1-\hat{p})/n} \\ &= 1.96 \cdot \sqrt{(.23)(.77)/100} \\ &= 0.082 \end{aligned}$$

- The 95% confidence interval:

0.23 ± 0.082 -OR- 0.148 to 0.312

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- In *any* normal distribution 95% of the distribution is within 2 standard deviations of the mean - actually it's closer to 1.96 standard deviations of the mean
- A single $\hat{p}$ calculated from a single sample of size n has probability 0.95 of being within 1.96 standard deviations of p!

Q: *Which of the following is the correct interpretation of the 95% confidence interval we computed for the yellow rectangle proportion? We are 95% confident that*

(a) *14.8% to 31.2% of all rectangles in my sample are yellow.*

(b) *14.8% to 31.2% of all rectangles in the population of all of Hitchman's rectangles are yellow.*

(c) *there is a 14.8% to 31.2% chance that a randomly chosen rectangle is yellow.*

(d) *there is a 14.8% to 31.2% chance that 95% of all the rectangles are yellow.*

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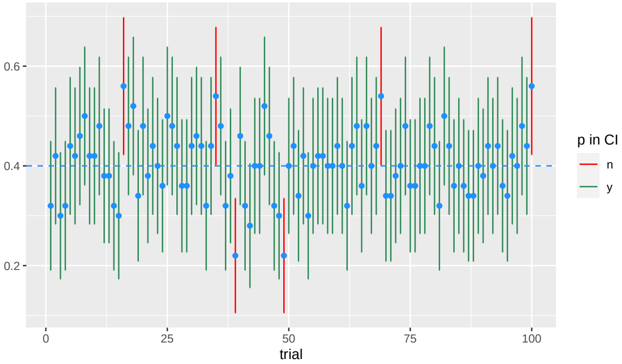
What does 95% confident mean?

- Suppose we took many samples of 100 rectangles, and built a confidence interval from each sample using the equation $point\ estimate \pm 1.96 \times SE$.
- Then about 95% of those intervals would contain the true population proportion (p).

What does 95% confident mean?

Simulation (100 repetitions): $p = 0.40$, determine a 95% confidence interval for p based on a sample of size $n = 50$.



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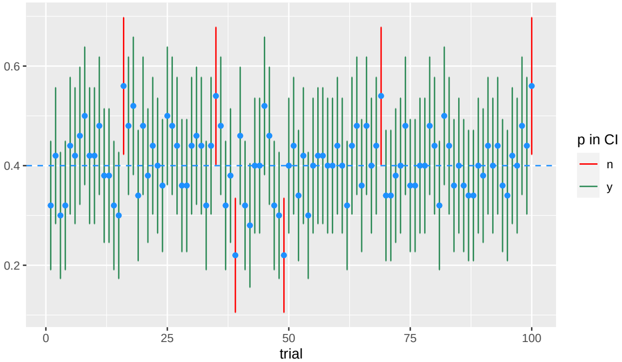

We see that 94 of the 100 confidence intervals in this simulation actually contain the population proportion $p = .4$.

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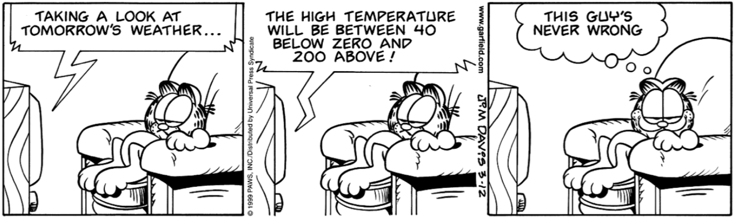

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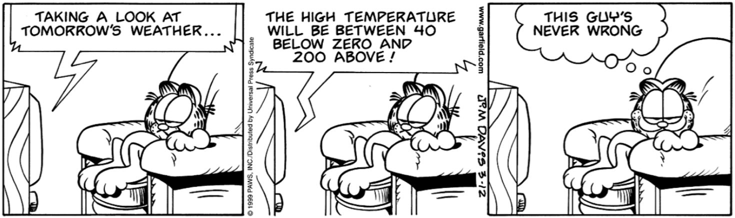

If the interval is too wide it may not be very informative.

Image source: http://web.as.uky.edu/statistics/users/earo227/misc/garfield_weather.gif

Changing the confidence level

$$point\ estimate \pm z^{\star} \times SE$$

- In a confidence interval, $z^{\star} \times SE$ is called the *margin of error*, and for a given sample, the margin of error changes as the confidence level changes.
- In order to change the confidence level we need to adjust $z^{\star}$ in the above formula.
- Commonly used confidence levels in practice are 90%, 95%, 98%, and 99%.
- For a 95% confidence interval, $z^{\star} = 1.96$.
- However, using the standard normal (z) distribution, it is possible to find the appropriate $z^{\star}$ for any confidence level.

Q: *Which of the below Z scores is the appropriate $z^\star$ when calculating a 98% confidence interval?*

(a) $Z = 2.05$

(b) $Z = 1.96$

(c) $Z = 2.33$

(d) $Z = -2.33$

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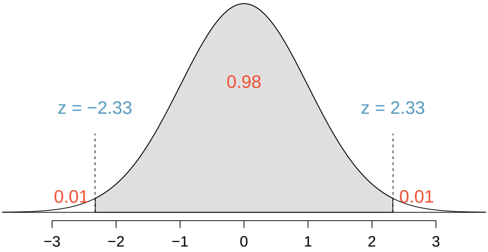
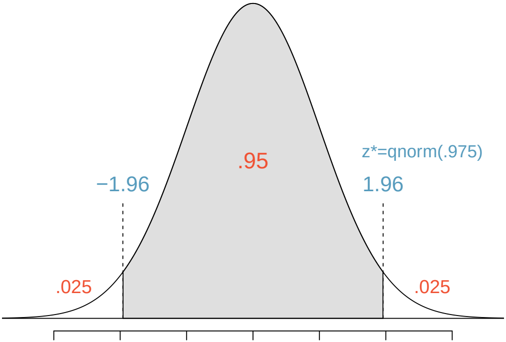


Finding z^* for a given confidence level

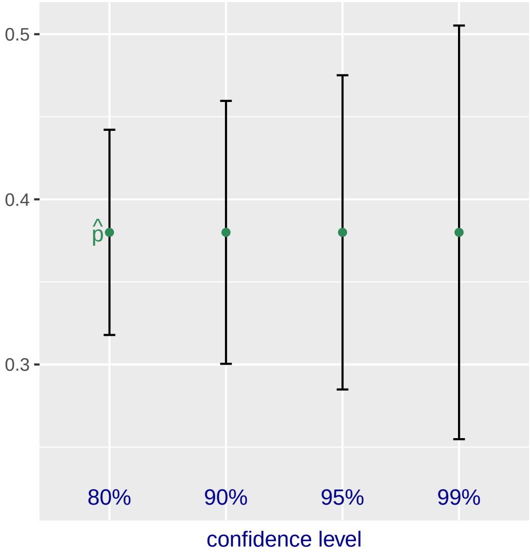
A level L confidence interval for a population proportion p

$$\hat{p} \pm z^* SE$$

$\hat{p}$ - sample proportion from an independent sample of size n
SE $= \sqrt{\hat{p}(1-\hat{p})/n}$, and $z^* = \texttt{qnorm}(L + (1-L)/2)$.



Confidence levels impact the size of the confidence interval



Interpreting confidence intervals

Confidence intervals are ...

- always about the population
- not probability statements
- only about population parameters, not individual observations
- only reliable if the sample statistic they're based on is an unbiased estimator of the population parameter

Confidence interval for a single proportion

Once you've determined a one-proportion confidence interval would be helpful for an application, there are four steps to constructing the interval:

- **Prepare.** Identify $\hat{p}$ and n, and determine what confidence level you wish to use.
- **Check.** Verify the conditions to ensure $\hat{p}$ is nearly normal. For one-proportion confidence intervals, use $\hat{p}$ in place of p to check the success-failure condition.
- **Calculate.** If the conditions hold, compute SE using $\hat{p}$, find $z^{\star}$, and construct the interval.
- **Conclude.** Interpret the confidence interval in the context of the problem.

Example: Solar Energy

In a Pew Research poll in 2018 about solar energy, 84.8% of respondents supported expanding the use of wind turbines (see text p. 186). Follow the four steps for creating a 99% confidence interval for the level of American support for expanding the use of wind turbines for power generation.

Example: Solar Energy

In a Pew Research poll in 2018 about solar energy, 84.8% of respondents supported expanding the use of wind turbines (see text p. 186). Follow the four steps for creating a 99% confidence interval for the level of American support for expanding the use of wind turbines for power generation.

Prepare. Here, we are being asked to use 99% confidence level, we have $n = 1000$ and $\hat{p} = .848$.

Example: Solar Energy

Check. The Pew survey was a random sample, and we can check that both counts (in favor and opposed) are at least 10:

$$n\hat{p} = 848 \text{ and } n(1 - \hat{p}) = 152.$$

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Calculate. $z^* = \texttt{qnorm}(.995) = 2.576$, SE $= \sqrt{\frac{.848(1-.848)}{1000}} = 0.0114$. So the margin of error MOE $= 2.576 \cdot 0.0114 = 0.0294$ and the 99% confidence interval is:

$$0.848 \pm 0.0294 \quad \text{or} \quad (0.8186, 0.8774).$$

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Conclude. We are 99% confident the proportion of American (in 2018) who support expanding the use of wind turbines is between 81.9% and 87.7%.

Choosing a sample size to obtain desired margin of error

The margin of error for a confidence interval is:

$$\text{MOE} = z^* \cdot \sqrt{\frac{p(1-p)}{n}}.$$

If we specify a confidence level, say 95%, then this determines z^*, ($z^* = 1.96$ in this case).
The other factor we control is our choice of sample size, n. As n increases, MOE decreases.

Example

Find the sample size n that will give a MOE of .03 in a 95% confidence interval, assuming $p = .5$.

Solution

Solve this equation for n:

$$.03 = 1.96 \cdot \sqrt{\frac{(.5)(.5)}{n}}$$

$$\frac{.03}{1.96} = \sqrt{\frac{.25}{n}}$$

$$\left(\frac{.03}{1.96}\right)^2 = \frac{.25}{n}$$

$$n = \frac{(.25)(1.96)^2}{(.03)^2} \approx 1067.11$$

Since n must be a whole number, we round up to 1068.

General Formula for specifying sample size n to produce desired MOE

We can solve the MOE formula for n we obtain the following general formula:

Finding sample size for desired MOE

$$n = p(1-p)\left(\frac{z^*}{\text{MOE}}\right)^2$$

If you don't know p, letting $p = 1/2$ to produce an n that will work regardless of the actual value of p, in which case the formula for n becomes

$$n = \frac{1}{4}\left(\frac{z^*}{\text{MOE}}\right)^2$$